Thermodynamic geometry and extremal black holes in string theory

Tapobrata Sarkar, Gautam Sengupta and Bhupendra Nath Tiwari

Department of Physics, Indian Institute of Technology,
Kanpur 208016, India
E-mail: `tapo@iitk.ac.in`, `sengupta@iitk.ac.in`, `bntiwari@iitk.ac.in`

ABSTRACT: We study a generalisation of thermodynamic geometry to degenerate quantum ground states at zero temperatures exemplified by charged extremal black holes in type II string theories. Several examples of extremal charged black holes with non degenerate thermodynamic geometries and regular but vanishingly small state space scalar curvatures are established. These include black holes described by D1-D5-P and D2-D6-NS5-P brane systems and also two charged small black holes in Type II string theories. We also explore the modifications to the state space geometry and the scalar curvature due to the higher derivative contributions and string loop corrections as well as an exact entropy expression from quantum information theory. Our construction describes state space geometries arising out of a possible limiting thermodynamic characterisation of degenerate quantum ground states at zero temperatures.

KEYWORDS: Black Holes in String Theory, D-branes.

Contents

1. Introduction

Over the past decade, black hole thermodynamics has emerged as a crucial theoretical laboratory to test issues of quantum gravity in the context of string theories. The area has witnessed major advances especially toward a resolution of the microscopic statistical basis underlying the macroscopic entropy of extremal and near extremal black holes in string theory [1]. Macroscopically black holes are known to be thermodynamic systems with a characteristic Hawking temperature and an entropy, which, upto leading order, is proportional to the area of the event horizon in Planck units [2]. The entropy is a function of the mass (internal energy) M, charge Q and the angular momentum J for the most general charged rotating black holes. These serve as extensive thermodynamic variables provided we consider the black hole to be a subsystem of a larger thermodynamic system with which it is in equilibrium.

A large class of extremal BPS black holes occur in the low energy supergravity theories arising from string theory. In particular the entropy of extremal black holes in supergravity theories with $N \geq 2$ are known to posses an underlying microscopic statistical description

in terms of D-brane systems or fundamental string states. Although such extremal black holes have zero Hawking temperature, they have a non zero thermodynamic entropy and are described by degenerate quantum ground states. A microscopic state counting in the associated conformal field theory then reproduces the thermodynamic entropy as an asymptotic expansion in the large charge limit. The low energy effective action of $N \geq 2$ supergravity following from type II string compactifications also involve higher derivative terms in an α' expansion. These terms modify the Bekenstein Hawking area law and introduces subleading corrections to the entropy. These corrections may be computed from the Wald formulation of generally covariant higher derivative theories of gravity [2]. It has been possible in the recent past to account for these subleading corrections from a microscopic perspective following from the underlying string theory. Exact matching between the macroscopic and microscopic entropy upto various subleading orders in an asymptotic expansion have been obtained for diverse extremal black holes [3].[1]

On the other hand, thermodynamic systems in equilibrium are known to possess interesting extrinsic geometrical features [4] although intrinsic geometric structures in equilibrium thermodynamics were largely unknown. An equilibrium state space of a thermodynamic system may be considered to be described by the minima of the internal energy function $U = U(S/T, V/T, \mu_i/T)$ in the energy representation or the maxima of the entropy $S = S(U, V, N_i)$ in the entropy representation. Here the quantities (μ_i, T, V, N_i) are the chemical potentials, temperature, volume and particle number per species respectively. Weinhold [5] introduced an inner product in this thermodynamic state space in the energy representation as the Hessian matrix of the internal energy with respect to the extensive variables leading to an intrinsic positive definite Riemannian geometric structure. One of the extensive parameters, typically the volume, was held fixed to provide a physical scale and prevent the development of negative eigenvectors of the metric. Although interesting, the physical relevance of this structure seemed elusive. Ruppeiner [6] reformulated the Weinhold inner product in the entropy representation in terms of the (negative of the) Hessian matrix of the entropy with respect to the extensive thermodynamic variables. This also led to a positive definite Riemannian geometric structure in the thermodynamic state space which was conformally related to the Weinhold geometry, with the temperature as the conformal factor. Ruppeiner showed that consideration of thermodynamic fluctuation theory [7] in addition to the thermodynamic laws allowed a remarkable physical interpretation of this geometric structure in terms of the probability distribution of the fluctuations and a relation of the scalar curvature with critical phenomena.

Thermodynamic geometry of the equilibrium state space described above may also be applied to study black holes considered as thermodynamic systems. Recent studies of the thermodynamics of diverse black holes in this geometric framework have elucidated interesting aspects of phase transitions and relations to moduli spaces of $N \geq 2$ supergravity compactifications in the context of extremal black hole solutions in these theories. It may be argued, however, that the connection of this formulation to fluctuation theory for

---

[1]There is a huge amount of literature in the subject we give here some of the recent lectures which summarizes the field.

application to black holes requires several modifications [9]. The geometric formulation in the thermodynamic state space was first applied to extremal black holes in $D = 4$, $N \geq 2$ supergravity which arise as low energy effective field theories from compactification of Type II string theories on Calabi-Yau manifolds [10]. A connection between the geometric formulation and the moduli space metric for the appropriate Calabi-Yau manifold could be established in this framework. Since then, several authors have attempted to understand the thermodyamic geometry of non extremal black holes [11, 12] and five dimensional rotating black rings. In this context, we had explored the state space geometry of both non extremal rotating BTZ black holes and rotating BTZ-Chern Simons (BTZ-CS) black holes in the Ruppeiner formulation [13].

The non zero entropy of extremal black holes and other such examples of degenerate quantum ground states indicates the possibility of a limiting characterization of conventional theromdynamics to such systems at zero temperatures. It is then an issue of importance to investigate whether the original domain of application of thermodynamic geometries may also be extended to such degenerate quantum ground states. Although such a possibility was alluded to in [10] there was no conclusive computation or elucidation of the issues arising from such an extension. Clearly, notions of conventional thermodynamics are not expected to be valid in this regime, and one would require modifications to the same. The first and most fundamental question that one may address, however, is whether a geometric characterization of the state space is at all possible at extremality. It is one of the issues that we will address in this paper, and our conclusion is that this is indeed the case. It should be emphasized that the issue of a non degenerate thermodynamic geometry at extremality is not at all obvious as conventional thermodynamic notions are invalid. As such, our results strongly suggest the possibility of a geometric characterization of black hole thermodynamics even at zero temperature, which was not known earlier. In fact our computations serve as the first examples of a geometric realization of the state space for degenerate quantum ground states exemplified by zero temperature extremal black holes.

In this paper, we will focus our attention on the charged extremal BPS black holes in Type II string theories compactified on Calabi-Yau manifolds. As is well known by now, the radial variation of the moduli in these cases exhibit attractor behaviour as they flow from their asymptotic values to an attractor fixed point at the horizon where they are fixed in terms of the charges through the attractor (stabilization) equations. The attractor mechanism could be described in terms of an *effective potential* $V(q, p, \phi)$ which was a function of the conserved charges and the scalar moduli and expressed in terms of certain symplectic invariants of the $N = 2$ special geomtery. The critical points of this effective potential at which its first derivative with respect to the moduli vanished were characterized as the attractor fixed points at which the central charge of the $N = 2$ supersymmetry also assumed its minimum value. At the attractor fixed point the macroscopic entropy was given as $S_{\mathrm{macro}} = \frac{A_h}{4} = \pi V(p, q, \phi_h^a)$, where ϕ_h^a were the fixed point values of the moduli and the entropy was purely a function of the charges alone. These BPS black hole solutions (of $N = 2$ supergravity) fall in two distinct classes; namely the large black holes which have a non vanishing area at the two derivative level and posses dyonic charges, and the small black holes which have a vanishing area and carry electric charges only in a suitable duality

basis. The large black holes may be described in terms of wrapped branes on non-trivial cycles of the compact internal manifold. Their microscopic entropy is determined in terms of the microstate counting through the Cardy formula in the underlying two dimensional CFT associated with the brane system. This is in precise agreement upto the subleading terms with the macroscopic entropy following from the Wald formula. The small black holes are more complicated. They have a vanishing horizon area as the horizon coincides with the null singularity and the curvature diverges. Hence higher curvature terms are large and the singularity is cloaked by the effective horizon [3]. Although a lot of progress has been made in understanding these systems, there are still certain unresolved issues.

The macroscopic entropy also follows directly from a variational principle applied to a generic class of *entropy functions* of the charges and the moduli. The attractor equations arise as a consequence of extremisation of this function with respect to the moduli and lead to the attractor fixed point at the horizon. An alternative analysis due to Sen [15] involves an adaptation of the Wald formalism to establish a more general variational technique to compute the higher derivative corrections to the entropy of charged extremal black holes. This formalism involves a more general class of entropy functions which are functions of both the scalar moduli and the parameters which describe the near horizon $AdS_2 \times S^{(D-2)}$ geometry. Extremisation of this entropy function determines all the near horizon parameters and it maybe shown that the entropy function at the attractor fixed point determines the black hole entropy. Higher derivative contributions to the entropy may also be elegantly implemented through this general entropy function formalism. Typically, the generalised entropy function formalism is mostly independent of supersymmetry considerations and has been also applied to extremal but non supersymmetric black holes [16].

As mentioned earlier the non zero entropy of extremal black holes which have zero Hawking temperatures naturally alludes to a non trivial limiting characterisation of conventional thermodynamics. This is further supported by the gauge-gravity correspondence in which certain limiting thermodynamic notions emerge for describing extremal black holes [17]. The entropy arises from macroscopic degeneracy of a quantum ground state and is a familiar phenomena in the physics of condensed matter systems like spin glasses. A thermodynamic interpretation of this macroscopic degeneracy for extremal black holes may be formally attempted through the partition function in a grand canonical ensemble involving summation over the chemical potentials. Although an exact evaluation of this formal expression is difficult it maybe possible within the gauge gravity correspondence to compute the sum in the boundary gauge theory [18]. An alternative approach to a limiting zero temperature characterisation of thermodynamics also arises from the general AdS-CFT correspondence [19, 20].

This naturally leads to the question of the possibility of a geometric characterisation of the equilibrium thermodynamic state space of degenerate quantum ground states such as extremal black holes. This is a reduced state space of equilibrium thermodynamic states consistent with the extremality condition. For supersymmetric black holes this is simply the BPS condition although the general arguments should also hold for non supersymmetric extremal black holes. As geometric notions remain valid even when conventional thermodynamics is invalid, a non degenerate geometric realization of the state space of

extremal black holes should be possible within the framework of thermodynamic geometries. In accordance with the first law of thermodynamics, the equilibrium state space for these extremal black holes would now also involve the scalar moduli at asymptotic infinity apart from the electric and the magnetic charges [22] as extensive thermodynamic variables. This, in general, leads to a curved equilibrium thermodynamic state space.

Although conventional thermodynamic notions breakdown at extremality, the geometric features of the state space should continue to be valid and well defined. However, the connection to thermodynamic fluctuations is elusive even for non extremal black holes [9] and is certainly not expected to hold in the zero temperature extremal limit. In fact its well known that classical fluctuations which have a thermal origin are absent at zero temperatures. The scalar curvature on the other hand may still indicate interactions, and divergences in the scalar curvature possibly allude to zero temperature quantum phase transitions amongst distinct vacua in the moduli space. In particular a generalized thermodynamic geometry of the equilibrium state space extended by the moduli variables should lead to insights into thermodynamics away from the attractor fixed point. Such an analysis should also provide an understanding of the attractor mechanism in terms of flows in the space of thermodynamic geometries and a geometric comprehension of the attractor fixed point.

As a first step toward this objective, it is necessary to explore the state space geometry for extremal black holes at the attractor fixed point. In particular, it is important to explore whether thermodynamic metrics are non degenerate at extremality and to establish the behavior of the scalar curvature. Such a construction would clearly elucidate the issue of thermodynamics at extremality and provide a geometrical realization of the equilibrium thermodynamic state space at zero temperatures. It would also be interesting to study the effect of the higher derivative contributions to the state space geometry and the scalar curvature. It is possible that the higher derivative corrections may induce a modification of the divergences in the scalar curvature in the state space and consequently that of possible quantum phase structures. Small black holes in Type II supergravity which have zero entropy at the two derivative level are particularly interesting in this connection as the thermodynamic geometry arises only from the higher derivative contributions to the entropy.

We emphasize here however that since we are considering a limiting characterization of thermodynamics arising from the macroscopic degeneracy of a quantum ground state, conventional thermodynamic notions will need to be modified. In particular the usual relation of the geometry of equilibrium state space with thermodynamic fluctuations is not expected to be valid at zero temperatures at which such fluctuations are absent. The characterization of the probability distribution of thermodynamic fluctuations in terms of an invariant positive definite Riemannian form over the equilibrium state space is thus not expected to hold for extremal black holes. So although we see the emergence of a non degenerate geometric structure of the state space through our construction, positive definiteness of the Riemannian form is not a strict criterion. However, the usual notion of thermodynamic stability in the canonical ensemble requiring the positivity of specific heats or compressibilities are still expected to hold. In fact, the invariant Riemannian form on the thermodynamic state space for such systems can be indefinite and may even be

sensitive to the higher derivative corrections to the entropy. This issue may be understood from the fact that the Weinhold geometry with respect to the scalar moduli given by the Hessian of the ADM mass is proportional to the moduli space metric at the attractor fixed point with the BPS mass being a proportionality constant [10]. For extremal black holes which we consider the ADM mass is equal to the macroscopic entropy. Hence the sign of the Hessian of the macroscopic entropy with respect to the conserved charges would clearly depend on the signature of the moduli space metric.

In this paper, we examine the geometry of the equilibrium thermodynamic state spaces of three different charged extremal black holes in Type II supergravity with 4 charges, 3 charges and 2 charged small black holes. These are described respectively by microscopic D2-D6-NS5-P and D1-D5-P brane systems and Type IIB string theory compactified on $K_3 \times T^2$. The paper is organised as follows. In the next section we provide a brief review of thermodynamic geometries of the equilibrium state spaces of thermodynamic systems and its relation to interactions and phase transitions both for two and higher dimensional thermodynamic state spaces. In section 3, we explore the thermodynamic geometry of the four charged extremal black hole arising from a microscopic configuration of D2-D6-NS5-P brane system and further study the modification to the thermodynamic geometry from higher derivative contributions to the entropy. In section 4, we investigate the state space geometry of the three charged extremal black hole arising from a D1-D5-P system both in $D = 5$ and $D = 10$ and also illustrate the effect of higher derivative contributions to the entropy on the state space geometry and its curvature. In section 5, we take up the interesting issue of two charged small black holes in Type IIB string theory compactified on $K_3 \times T^2$ and explore the thermodynamic geometry of the state space arising out of both the macroscopic and the microscopic entropy expression resulting from a heterotic string theory computation. We further consider the modification of the state space geometry due to the higher derivative corrections to the entropy both from the low energy effective action as well as string loop corrections. We also study the state space geometry and its curvature implied by an exact entropy expression following from a quantum information theory perspective. Section 6 concludes this paper with a summary of the results.

2. Review of thermodynamic geometry

In this section we present a brief review of the essential features of thermodynamic geometries and their application to the thermodynamics of black holes, in particular extremal black holes. This will serve to to set the notations and conventions used in the rest of this paper. An intrinsically geometric structure in equilibrium thermodynamics was introduced by Weinhold [5] through an inner product in the space of equilibrium thermodynamic macrostates defined by the minima of the internal energy function $U = U(S/T, V/T, \mu_i/T)$ as the Hessian

$$h_{ij} = \partial_i \partial_j U \tag{2.1}$$

As mentioned in the introduction, the quantities (μ_i, T, V, S) are the chemical potentials, temperature, volume and entropy respectively and the volume or any other parameter is

held fixed to provide a physical scale and to restrict negative eigenvectors of the metric. Although such a Riemannian geometric structure was interesting, no physical significance could be ascribed to it. The inner product on the state space was later reformulated by Ruppeiner [6] in the entropy representation as the negative of the Hessian matrix of the entropy with respect to the extensive variables. The thermodynamic macrostates underlying the equilibrium state space being now described by the maxima of the entropy function $S = S(U, V, N)$. Explicitly the Ruppeiner metric in the state space was given as

$$g_{ij} = -\partial_i \partial_j S(U, V, N) \qquad (2.2)$$

and was conformal to the Weinhold metric with the inverse temperature as the conformal factor. The negative sign was necessary to ensure positive definiteness of the metric, as the entropy is a maximum in the equilibrium state. It could be shown that the Riemannian structure defined by the Ruppeiner metric was closely related to classical thermodynamic fluctuation theory [8] and critical phenomena. The probability distribution of thermodynamic fluctuations in the equilibrium state space was characterised by the invariant interval of the corresponding thermodynamic geometry in the Gaussian approximation as

$$W(x) = A\exp\left[-\frac{1}{2}g_{ij}(x)dx^i dx^j\right] \qquad (2.3)$$

where A is a constant. The inverse metric may be shown to be the second moment of fluctuations or the pair correlation functions and given as $g^{ij} = < X^i X^j >$ where X^i are the intensive thermodynamic variables conjugate to x^i. The Riemannian structure could be expressed in terms of any suitable thermodynamic potential arrived at by Legendre transforms which corresponds to general coordinate transformations of the equilibrium state space metric. The geometric formulation tacitly involves a statistical basis in terms of a canonical ensemble although the analysis was considered only in the thermodynamic limit.

For a standard two dimensional thermodynamic state space defined by the extensive variables (x^1, x^2), application of these geometric notions to conventional thermodynamic systems suggest that a non zero scalar curvature indicates an underlying interacting statistical system. It may be shown that the scalar curvature $R \sim \kappa_2 \xi^d$ where ξ is the correlation length, d is the physical dimensionality of the system and κ_2 is a dimensionless constant of order one. Hence the scalar curvature diverges at the critical points. The Ruppeiner formalism has been applied to diverse condensed matter systems with two dimensional state spaces and is completely consistent with the scaling and hyperscaling relations involving critical phenomena and have reproduced the corresponding critical indices.

Having provided a brief account of thermodynamic geometries, in the following sections we systematically explore the state space geometry in the Ruppeiner framework for several charged extremal black holes in Type II string theories. Our constructions would provide a geometric realization of a limiting equilibrium thermodynamics at zero temperatures. This would be the first step to address the issue of the thermodynamics of extremal black holes away from the attractor fixed point and a geometric characterization of the attractor mechanism. It will also serve as a prelude to the application of the formalism

of thermodynamic geometries to the study of zero temperature quantum phase transitions amongst distinct vacua in the moduli space of string theory compactifications.

The application of this geometric formalism to non extremal black holes and the consequent divergences of the scalar curvature at known critical points [12] indicates that a non zero scalar curvature for state spaces of extremal black holes may also suggest an underlying interacting statistical system. In this case, the divergences of the scalar curvature may allude to zero temperature quantum phase transitions amongst distinct vacua in the moduli space. The attractor mechanism for extremal black holes and the consequent flow of the scalar moduli to fixed values at the horizon in terms of the charges also seems to suggest such a connection of the state space scalar curvature with the structure of the moduli space. It is this perspective that we will adopt in the present study and we explore the scalar curvatures over the state space of extremal black holes and their sensitivity to higher derivative corrections to the entropy. Following the arguments presented earlier we will interpret the scalar curvature as indicative of an interacting microscopic statistical system underlying extremal black holes at zero temperatures.

3. Four charged black holes in D2-D6-NS5-P system

In this section, as a first exercise, we study the thermodynamic geometry of four charged extremal black holes in $D = 4$ and $D = 10$ in Type II A string theory compactified on T^6 [23]. These are known to be described by microscopic D2-D6-NS5-P brane systems which are $\frac{1}{8}$ BPS configuration in Type IIA supergravity. The solitonic NS5 brane is required for satisfying the tree level IIA supergravity equations of motion and this does not affect the overall supersymmetry. From a microscopic perspective this is analogous to exciting left moving oscillations on a fundamental heterotic string. The brane system in question is T dual to the D1-D5-P system considered in the next section. The D2 branes are sources for electric fields whilst the D6 and NS5 branes are sources of magnetic fields. For the relevant expressions for the metrics in the various cases, see, e.g. [24].

3.1 Four charged black holes in $D = 4$

The macroscopic entropy at the tree level $\alpha\prime$ resulting from the two derivative part of the action is

$$S(N_2, N_5, N_6, N_p) = 2\pi \sqrt{N_2 N_5 N_6 N_p} \tag{3.1}$$

where, in the D-brane description, N_2, N_6 and N_5 are identified with the number of D-2, D-6 and NS-5 branes respectively, and N_p with the number of units of Kaluza Klein momenta, and are all assumed to be large. The equation is thus valid in the limit of large charges and small curvature. The entropy is the standard Bekenstein-Hawking entropy given by the area law. It is now possible to explore the thermodynamic geometry of the equilibrium state space of the 4 charged extremal black hole in $D = 4$ arising from this entropy expression. The Ruppeiner metric in the state space is given by the Hessian matrix of the entropy with respect to the extensive variables which in this case are the four conserved charges carried

by the extremal black hole. A straightforward computation yields

$$ds^2 = \frac{\pi}{2} \left(\sqrt{\frac{N_5 N_6 N_p}{N_2}} \frac{dN_2^2}{N_2} + \sqrt{\frac{N_2 N_6 N_p}{N_5}} \frac{dN_5^2}{N_5} + \sqrt{\frac{N_2 N_5 N_p}{N_6}} \frac{dN_6^2}{N_6} + \sqrt{\frac{N_2 N_5 N_6}{N_p}} \frac{dN_p^2}{N_p} \right)$$
$$- \pi \left(\sqrt{\frac{N_6 N_p}{N_2 N_5}} dN_2 dN_5 + \sqrt{\frac{N_5 N_P}{N_2 N_6}} dN_2 dN_6 + \sqrt{\frac{N_5 N_6}{N_2 N_p}} dN_2 dN_p \right.$$
$$\left. + \sqrt{\frac{N_2 N_p}{N_5 N_6}} dN_5 dN_6 + \sqrt{\frac{N_2 N_6}{N_5 N_p}} dN_5 dN_p + \sqrt{\frac{N_2 N_5}{N_6 N_p}} dN_6 dN_p \right) \tag{3.2}$$

The determinant of the metric tensor is $g = -\pi^4$ which is a constant and hence we have a non degenerate metric for the thermodynamic state space at extremality. Note that the determinant is of negative sign, implying that the metric is *not* positive definite. As we have emphasized that in the case of extremal black holes being considered, the connection between the positivity of the thermodynamic metric and classical fluctuations of a thermal origin is invalid, as such fluctuations cease at zero temperatures. However this does not rule out quantum fluctutations. We also reiterate that the relation of the Hessian of the *effective potential* with respect to the moduli and the moduli space metric [10] at the attractor fixed point, clearly indicates that the signature of the Ruppeiner metric depends on that of the moduli space metric.

The fact that the metric is non-degenerate is somewhat remarkable, as conventional thermodynamics breaks down at extremality. However our results indicate that a limiting characterisation of conventional thermodynamics at zero temperature should still hold at extremality. This limiting thermodynamic description would correspond to the characterisation of extremal black holes as degenerate quantum ground state exhibiting macroscopic degeneracy. The non degenerate metric at extremality thus provides for the first time a geometric realization of the thermodynamic state space of the extremal black hole at zero temperature.

Note that the state space geometry is four dimensional as the thermodynamic entropy is a function of four charges which serve as extensive variables. As mentioned earlier, classical fluctuations are absent at zero temperatures although the curvature scalar should still continue to indicate an underlying interacting statistical system given the fact that it does so for non extremal black holes. The curvature scalar over the state space in this case may be easily computed to be

$$R = \frac{3}{2\pi \sqrt{N_2 N_5 N_6 N_p}} \tag{3.3}$$

Thus, the curvature scalar is non zero, finite and regular everywhere although in the large charge limit at which the entropy computation is valid the curvature is vanishingly small. This should indicate an underlying non interacting and stable microscopic statistical basis. The entropy considered here is the one arising from the usual two derivative terms in the low energy effective supergravity action which is consistent with the area law. This is modified by contributions from the higher derivative terms in the effective action, and consequently modifies the equilibrium state space and the thermodynamic geometry including

the curvature scalar. In what follows we will consider such modifications to the entropy of the $D = 10$ four charged black holes and study the consequent thermodynamic geometries of the equilibrium state space based on the modified entropy expression.

3.2 Four charged black holes in $D = 10$

The near horizon geometry of the $D = 10$ extremal charged black holes described by the D2-D6-NS5-P system is $AdS_3 \times S^2 \times S^1 \times T^4$. The Wald formula in the framework of the Sen entropy function with some modifications for the non standard near horizon geometry leads to the entropy of the four charged extremal black hole in $D = 10$ at the two derivative level as

$$S(N_2, N_5, N_6, N_p) = 2\pi\sqrt{N_2 N_5 N_6 N_p} \qquad (3.4)$$

with the N_is defined as before. This is identical to the $D = 4$ case and follows the standard area law and the universal expression for entropy of charged extremal black holes as the square root of the product of the charges. The thermodynamic geometry of the equilibrium state space arising from the entropy at the two derivative level is hence also identical to the $D = 4$ case.

It is possible to consider the subleading corrections to the entropy of the four charged extremal black hole in $D = 10$ following from the contribution of the higher derivative terms in the low energy effective action. The Sen entropy function framework is applicable for this exercise in spite of the non standard horizon geometry, with certain modifications. The corrected entropy actually depends on a parameter which involves field redefinitions for the higher derivative terms [24]. This essentially happens for the D2-D6-NS5-P system because the supergravity configuration admits a near horizon geometry involving AdS_3 and S^2 with different radii (unlike the D1-D5-P system considered later). The ambiguity due to field redefinitions may be used to define a specific redefinition scheme in which only the Weyl tensor part of the curvature occurs in the higher derivative terms. The $\alpha\prime$ corrected entropy for the four charged extremal black hole may now be computed using the Wald formula and the Sen entropy function extremisation framework. An explicit computation gives the corrected entropy as [24]

$$S(N_2, N_5, N_6, N_p) = 2\pi\sqrt{N_2 N_5 N_6 N_p} - 2\pi C \frac{\sqrt{N_p}}{N_2 N_6 N_5^{5/2}} \qquad (3.5)$$

where $C = \frac{73315}{222184}\left(\frac{2R_4}{G_N^4 \alpha\prime R_9}\right)^{3/2}\gamma$ where R_4 and R_9 refer to the radii of the circles over which the D-2 brane is wrapped, G_N^4 is the 4-D Newton's constant, and $\gamma = \frac{1}{8}\zeta(3)\alpha\prime^3$ The thermodynamic geometry of the equilibrium state space for the four charged extremal black hole resulting from the corrected entropy may now be computed from the Hessian matrix of the entropy with respect to the charges as:

$$ds^2 = \left(\frac{\pi}{2N_2}\sqrt{\frac{N_5 N_6 N_p}{N_2}} + \frac{4\pi C\sqrt{N_p}}{N_2^3 N_6 N_5^{5/2}}\right) dN_2^2 - \left(\pi\sqrt{\frac{N_6 N_p}{N_2 N_5}} - \frac{10\pi C\sqrt{N_p}}{N_2^2 N_6 N_5^{7/2}}\right) dN_2 dN_5$$

$$- \left(\pi\sqrt{\frac{N_5 N_P}{N_2 N_6}} - \frac{4\pi C\sqrt{N_p}}{N_2^2 N_6^2 N_5^{5/2}}\right) dN_2 dN_6 - \left(\pi\sqrt{\frac{N_5 N_6}{N_2 N_p}} + \frac{\pi C}{\sqrt{N_p} N_2^2 N_6 N_5^{5/2}}\right) dN_2 dN_p$$

$$+ \left(\frac{\pi}{2N_5} \sqrt{\frac{N_2 N_6 N_p}{N_5}} + \frac{35\pi C \sqrt{N_p}}{2N_2 N_6 N_5^{9/2}} \right) dN_5^2 - \left(\pi \sqrt{\frac{N_2 N_p}{N_5 N_6}} - \frac{10\pi C \sqrt{N_p}}{N_2^2 N_6 N_5^{7/2}} \right) dN_5 dN_6$$

$$- \left(\pi \sqrt{\frac{N_2 N_6}{N_5 N_p}} + \frac{5\pi C}{\sqrt{N_p} N_2 N_6 N_5^{7/2}} \right) dN_5 dN_p + \left(\frac{\pi}{2N_6} \sqrt{\frac{N_2 N_5 N_p}{N_6}} + \frac{4\pi C \sqrt{N_p}}{N_2 N_6^3 N_5^{5/2}} \right) dN_6^2$$

$$- \left(\pi \sqrt{\frac{N_2 N_5}{N_6 N_p}} + \frac{2\pi C}{\sqrt{N_p} N_2 N_6^2 N_5^{5/2}} \right) dN_6 dN_p$$

$$+ \left(\frac{\pi}{2N_p} \sqrt{\frac{N_2 N_5 N_6}{N_p}} - \frac{\pi C}{2N_p^{3/2} N_2 N_6 N_5^{5/2}} \right) dN_p^2 \tag{3.6}$$

The determinant of the metric tensor is

$$g = \frac{-\pi^4}{N_2^6 \sqrt{N_p} N_5^{12} N_6^6} (6C^2 \sqrt{N_p} N_2^3 N_5^6 N_6^3 + 5C N_2^4 N_5^{17/2} N_6^4 \sqrt{N_2 N_5 N_6 N_p}$$

$$+ 50 C N_2 N_5^{5/2} N_6 \sqrt{N_2 N_5 N_6 N_p} + N_2^6 \sqrt{N_p} N_5^{12} N_6^6 + 100 C^4 \sqrt{N_p}) \tag{3.7}$$

and is non zero in the large charge limit indicating a non degenerate thermodynamic geometry at extremality modified by the higher derivative contributions. Notice however that the determinant is negative suggesting that the signature of the state space metric is indefinite. As we have argued earlier this arises from the connection of the thermodynamic geometry to the moduli space metric [10]. The scalar curvature of the thermodynamic state space may now be computed in a straightforward fashion. The explicit expression for the scalar curvature is long and is hence relegated to the appendix. The scalar curvature is regular in the large charge limit in which the asymptotic expansion for the entropy is valid. We provide a graphical analysis of the scalar curvature for different values of the charges in figure 1. In the figure, we have shown the Ruppeiner scalar curvature of the D2-D6-NS5-P system, as a function of N_2 (X axis) and N_5 (Y axis), both of which vary from 10^3 to 3×10^3. To illustrate the example, we have chosen the typical values $N_6 \simeq N_p = 2 \times 10^3$. The value of the constant C has been set to 10^{-5}. We notice a small but positive scalar curvature in the state space at these typical values of N_2 and N_5.

4. Black holes in the D1-D5-P system

In this section we explore the thermodynamic geometry of 3 charged extremal black holes in Type IIB string theory arising from a microscopic configuration of N_1 D1 branes wrapped along some compact direction y with radius R, N_5 D5 branes wrapped along y and a four torus T^4, and $p = N_p/R$ units of KK momentum along the y direction. The corresponding five dimensional extremal black holes carry both electric and magnetic charges arising from the D1 and D5 branes respectively. In $D = 10$ the near horizon geometry of the configuration is $M_3 \times S^3 \times T^4$ where M_3 is a boosted AdS_3 geometry. The ten dimensional metric may be reduced over $S^1 \times T^4$ and the resulting $D = 5$ charged extremal black hole involves a more tractable near horizon geometry of $AdS_2 \times S^3$. The near horizon geometries imply that the standard entropy function method may be applied to compute
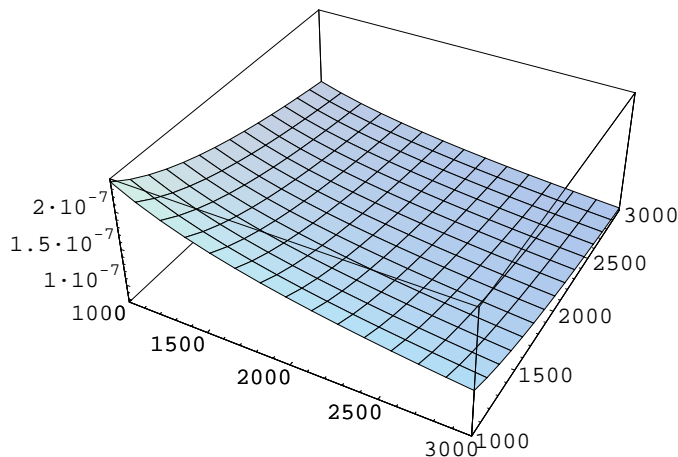
Figure 1: Ruppeiner curvature for the D2-D6-NS5-P System as a function of N_2 (X-axis) and N_5 (Y-axis).

the entropy of these extremal charged black holes at the two derivative level. Higher derivative contributions are however tricky as the non standard near horizon geometry for the corresponding configuration in $D = 10$ complicates the application of the Wald formula. Despite this the $\alpha^{3\prime} R^4$ corrections arising from string tree level scattering amplitudes have been incorporated to compute the subleading corrections to the entropy of extremal charged black holes in such D1-D5-P systems in $D = 10$ [25].

We now explore the thermodynamic geometry of these D1-D5-P extremal charged black holes in $D = 5$ and $D = 10$. We further elucidate the modification of the thermodynamic geometry involving the higher order $\alpha^{3\prime} R^4$ corrections in the $D = 10$ type IIB supergravity effective action, and their subleading contributions to the entropy

4.1 D1-D5-P black holes in $D = 5$

The $D = 10$ IIB supergravity may be compactified to $D = 5$ on $S^1 \times T^4$. The near horizon limit of the metric is $AdS_2 \times S^3$ allowing the application of the entropy function formalism to compute the Bekenstein-Hawking entropy as

$$S_{BH} = 2\pi \sqrt{N_1 N_5 N_p}. \tag{4.1}$$

The result coincides with that obtained by direct computation of the horizon area verifying the usual area law at the two derivative level.

The metric of the thermodynamic state space in the entropy representation may now be obtained as before from the Hessian matrix of the entropy with respect to all the extensive thermodynamic variables which in this case are just the D1, D5 and P charges. Explicitly

the metric is given as

$$ds^2 = \frac{\pi}{2} \left(\sqrt{\frac{N_5 N_p}{N_1}} \frac{dN_1^2}{N_1} + \sqrt{\frac{N_1 N_p}{N_5}} \frac{dN_5^2}{N_5} + \sqrt{\frac{N_1 N_5}{N_p}} \frac{dN_p^2}{N_p} \right)$$
$$- \pi \left(\sqrt{\frac{N_p}{N_1 N_5}} dN_1 dN_5 + \sqrt{\frac{N_5}{N_1 N_p}} dN_1 dN_p + \sqrt{\frac{N_1}{N_5 N_p}} dN_5 dN_p \right) \qquad (4.2)$$

We observe that as in the previous example the Ruppeiner metric is non degenerate and regular everywhere even at extremality. The determinant of the metric tensor is

$$g = -\frac{\pi^3}{2\sqrt{N_1 N_5 N_p}} \qquad (4.3)$$

and is non zero for non zero charges.[2] This lends credence to the contention of a limiting characterisation of conventional thermodynamics at zero temperature, in this case. The Ruppeiner metric at extremality obtained by us thus characterises a geometric realization of the corresponding equilibrium thermodynamic state space at extremality.

The scalar curvature corresponding to the geometry of the thermodynamic state space may now be determined to be

$$R = \frac{3}{4\pi \sqrt{N_1 N_5 N_p}} \qquad (4.4)$$

We observe that the scalar curvature is non zero, positive and regular everywhere in the large charge small curvature limit at which the entropy expression is valid. The state space curvature is vanishingly small in the large charge limit. As mentioned in connection with the previous example of the four charged extremal black holes in Type II supergravity, this fact seems to be universal and is related to the typical form for the Ruppeiner geometry as the Hessian matrix of the entropy. The usual connection of Ruppeiner geometry with thermodynamic fluctuation theory needs to be modified in connection with the application to black holes [9]. However the standard interpretation of scalar curvature of the state space geometry to describe interactions in the underlying statistical system should continue to hold for non extremal black holes. In particular this should also hold at extremality and hence the vanishingly small scalar curvature for the state space of D1-D5-P system in the large charge limit should indicate an underlying non interacting and stable statistical basis.

4.2 D1-D5-P black holes in $D = 10$

The black hole solution corresponding to a microscopic D1-D5-P brane system maybe described by a modified non standard near horizon $AdS_3 \times S^3 \times T^4$ geometry. For this case, the horizon at $r = 0$ has a geometry of $S^1 \times S^3 \times T^4$. This complicates the application of the Sen entropy function approach to compute the black hole entropy. The relations between the number of D-branes with constants of integration and the ADM momentum in y needs to be computed again due to the modified near horizon geometry. Furthermore

---

[2]As in the previous subsection, the metric is not positive definite, but our previous argument will still hold in this case.

the Wald formula for this modified background needs to be recomputed and is involved due to the presence of an off diagonal term in the metric describing the modified near horizon geometry. Fortunately the computation for the Wald formula may be simplified by using certain properties for the Riemann tensor of the modified near horizon geometry. This enables the application of the Sen entropy function method and the entropy may be computed to leading order or two derivative level to be [25]

$$S(N_1, N_5, N_P) = 2\pi\sqrt{N_1 N_5 N_P} \tag{4.5}$$

which is exactly the same as in the $D = 5$ case. Hence the Ruppeiner geometry of the thermodynamic state space for the $D = 10$ D1-D5-P black hole is identical to the $D = 5$ case. The scalar curvature of the state space is also identical and exactly the same interpretations should continue to hold as in the $D = 5$ case.

As in the instance of the D2-D6-NS5 brane system, the entropy expression for the system in question would also be modified by the subleading contributions from the higher derivative terms in the low energy Type IIB supergravity action. Following exactly the same approach, at the next leading order in α' (with spacetime R^4 corrections) arising out of the string three loop scattering amplitudes, the entropy of the extremal black hole is given as

$$S_{BH} = 2\pi\sqrt{N_1 N_5 N_p}\left[1 + \frac{b}{(N_1 N_5)^{3/2}}\right] \tag{4.6}$$

where $b = 3\gamma\left[\frac{2\pi^3 V_4}{16\pi G_{10}}\right]^{3/2}$ with G_{10} being the 10 dimensional Newton's coupling constant, V_4 the volume of the T^4 and $\gamma = \frac{1}{8}\zeta(3)\alpha'^3$.

The Ruppeiner metric may now be computed as the Hessian matrix of the corrected entropy expression in eq. (4.6) to be,

$$\begin{aligned}
ds^2 =\ & \left[\frac{\pi}{2N_1}\sqrt{\frac{N_5 N_p}{N_1}}\left(1 + \frac{b}{(N_1 N_5)^{3/2}}\right) - \frac{9\pi b\sqrt{N_p}}{2N_1^3 N_5}\right] dN_1^2 \\
& - \left[\frac{\pi N_p}{\sqrt{N_1 N_5 N_p}}\left(1 + \frac{b}{(N_1 N_5)^{3/2}}\right) + \frac{3\pi b\sqrt{N_p}}{(N_1 N_5)^2}\right] dN_1 dN_5 \\
& - \left[\frac{\pi N_5}{\sqrt{N_1 N_5 N_p}}\left(1 + \frac{b}{(N_1 N_5)^{3/2}}\right) - \frac{3\pi b}{\sqrt{N_p} N_1^2 N_5}\right] dN_1 dN_p \\
& + \left[\frac{\pi}{2N_5}\sqrt{\frac{N_1 N_p}{N_5}}\left(1 + \frac{b}{(N_1 N_5)^{3/2}}\right) - \frac{9\pi b\sqrt{N_p}}{2N_1 N_5^3}\right] dN_5^2 \\
& - \left[\frac{\pi N_1}{\sqrt{N_1 N_5 N_p}}\left(1 + \frac{b}{(N_1 N_5)^{3/2}}\right) - \frac{3\pi b}{\sqrt{N_p} N_1 N_5^2}\right] dN_5 dN_p \\
& + \frac{\pi}{2N_p}\sqrt{\frac{N_1 N_5}{N_p}}\left(1 + \frac{b}{(N_1 N_5)^{3/2}}\right) dN_p^2
\end{aligned} \tag{4.7}$$

The determinant of the metric tensor is given as

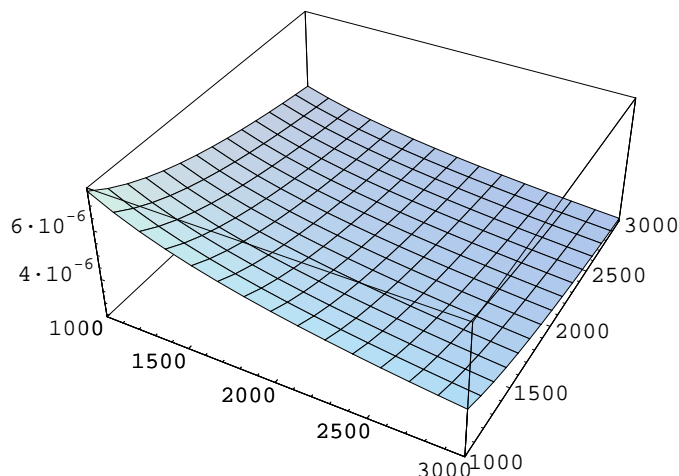$$g = -\frac{\pi^3}{2N_1^5 N_5^5 \sqrt{N_p}}\left[(N_1 N_5)^{9/2} + 6b(N_1 N_5)^{3/2} - 20b^3\right] \tag{4.8}$$

Figure 2: Ruppeiner curvature for the D1-D5-P system with higher derivative corrections, as a function of N_1 and N_5.

The determinant is non zero for non zero charges and as before we obtain a non degenerate thermodynamic geometry of the state space of the three charged extremal black hole at extremality. Note that as in the other examples the determinant is negative suggesting an indefinite signature of the Riemannian line element. Our earlier arguments that this arises from the connection with the signtaure of the moduli space metric remains valid.

The scalar curvature of the state space may now be computed using the Ruppeiner metric based on the Hessian matrix of the entropy corrected by higher derivative contributions. The curvature is regular everywhere although vansisgingly small in the large charge limit indicating again a stable thermodynamic system and underlying non interacting statistical basis. The exact expression for the scalar curvature is involved and we relegate it to the appendix. We trace the behavior of the scalar curvature graphically in figure 2, where we have plotted the Ruppeiner curvature as a function of N_1 and N_5, We have set the typical values $N_p = 10^3$ and $b = 10^{-5}$. Of course, the scalar curvature is identical to that of the leading order case for $b = 0$ for which the higher derivative contributions are absent.

5. Two charged small black holes

In this section we study the third example of Ruppeiner geometry of the thermodynamic state space of extremal two charged *small* black holes in string theory. These black holes are characterised by a vanishing horizon area and hence zero entropy at the two derivative level. However, higher derivative contributions renders this entropy to be non zero at the next to leading orders and also provides a finite non zero horizon area. Hence the naked singularity at the two derivative level is cloaked by a horizon arising from the higher derivative terms in the low energy effective supergravity action [3]. This renders the macroscopic description of small black holes somewhat complicated as the higher derivative terms are now crucial for the formation of a horizon. The microscopic statistical description of small black holes are however simpler as it is based on just the fundamental string states.

5.1 Macroscopic and microscopic description of small black holes

In Type IIA string theory compactified on $K_3 \times T^2$, small black holes are charged objects in a suitably chosen duality basis. The macroscopic entropy in the two charged example of small black holes is

$$S_{\text{macro}} = 4\pi \sqrt{|q_0 p^1|} \tag{5.1}$$

where q_0 and p^1 are the electric and magnetic charges respectively. In the dual description, this small black hole is a charged black hole with charges (q_0, p^1) of $N = 4$ heterotic string theory compactified on the six torus. It turns out that in this case the radius of the horizon is of the string length l_s and the small black hole is at the *correspondence point* where it may be described by a perturbative heterotic string state with the appropriate charge configuration.[3] It is possible to compute the degeneracy of these string states as the black hole is a $\frac{1}{2}$ BPS configuration and hence corresponds to a short multiplet of the $N = 4$ supersymmetry. The degeneracy may be computed by using the Rademacher formula and in the limit of large charges yields the microscopic entropy as

$$S_{\text{micro}} = \ln d(q, p) \simeq 4\pi \sqrt{|qp|} - 27/4 \ln |qp| + O\left(\frac{1}{\sqrt{|qp|}}\right) \tag{5.2}$$

$$\sim 4\pi (s + \overline{s})_{Hor.} - 27/2 \ln (s + \overline{s})_{Hor} \tag{5.3}$$

where s is the heterotic axion-dilaton field.

The macroscopic entropy matches the microscopic one at leading orders but fails at subleading orders where the microscopic entropy expression involves logarithmic corrections. So it follows that the macroscopic entropy also must incorporate such corrections. It turns out that non holomorphic corrections are required to be incorporated to S_{macro} as the microscopic entropy expression involves both s and its complex conjugate $\overline{s}$.

The toroidally compactified heterotic string theory posses a duality invariant spectrum but the entropy expression involving the generalized holomorphic prepotential of the $N = 2$ Lagrangian applied to toroidally compactified Type II string theory is not duality invariant. The duality invariance requires incorporation of non holomorphic terms to the generalized prepotential. After incorporating the same, in both the attractor equations and the macroscopic entropy, an S duality invariant expression is obtained which also incorporates a logarithmic correction term $-12 \ln(s + \overline{s})_{\text{Hor}}$. Thus, the mismatch between S_{macro} and S_{micro} persists even with the incorporation of the non holomorphic corrections. It could be shown later [16] that a computation in the heterotic description involving a grand canonical ensemble shows an exact match between the microscopic and the macroscopic entropy, although a Type II description still seems to be problematic.

5.2 Thermodynamic geometry of small black holes

The thermodynamic geometry of the equilibrium state space of two charged small black holes in Type IIA string theory may now be constructed based on the expressions for the microscopic and the macroscopic entropy. The horizon area for small black holes vanishes

---

[3]We will use the notation $q_0 = q$ and $p^1 = p$ in what follows

at the two derivative level and hence the leading order contribution to the macroscopic entropy is zero. The first contribution to the macroscopic entropy hence arises at the subleading order from the higher derivative contributions as

$$S_1 = 4\pi\sqrt{qp} \tag{5.4}$$

where q and p refer to the non-zero electric and magnetic charge of the black hole. Now, the Ruppeiner metric may be computed from the Hessian matrix of the entropy with respect to the charges. The equilibrium thermodynamic state space in this case is two dimensional as we have chosen a two charge configuration. Explicitly the metric is given as

$$ds^2 = \frac{\pi}{q}\sqrt{\frac{p}{q}}dq^2 - \frac{\pi}{\sqrt{qp}}dqdp + \frac{\pi}{p}\sqrt{\frac{q}{p}}dp^2 \tag{5.5}$$

It can be seen that the Ruppeiner metric based on the macroscopic entropy at the first subleading order, is degenerate and the determinant of the metric $g = 0$. Hence there is no viable thermodynamic geometry at this order for the two charged extremal small black hole in question.

Surprisingly the situation changes when we consider the duality invariant macroscopic entropy corrected by the non holomorphic contributions from the generalized prepotential. The macroscopic entropy in this case is given as

$$S(q,p) = 4\pi\sqrt{qp} - 6\ln(qp) \tag{5.6}$$

and the Ruppeiner metric of the state space may be easily computed from Hessian matrix of the entropy as

$$ds^2 = \left(\frac{\pi}{q}\sqrt{\frac{p}{q}} - \frac{6}{q^2}\right)dq^2 - \frac{2\pi}{\sqrt{qp}}dqdp + \left(\frac{\pi}{p}\sqrt{\frac{q}{p}} - \frac{6}{p^2}\right)dp^2 \tag{5.7}$$

The determinant of the metric is $g = -\frac{12}{q^2p^2}(\pi\sqrt{qp}-3)$. Hence, the thermodynamic geometry of the state space based on the above metric is non degenerate in the large charge limit under consideration as $g \neq 0$ in this limit. The determinant is negative according to the arguments presented for the earlier examples. The fact that a viable and non degenerate thermodynamic metric emerges only from a duality invariant entropy corrected by the non holomorphic contributions suggests a connection between duality invariant entropy and thermodynamic geometry at extremality. The theoretical issues of this connection is important to elucidate and it is expected that investigations of the state space geometry away from the attractor fixed point and relation to the moduli space geometry should clarify this connection further. We leave this issue for the future.

It is now straightforward to compute the scalar curvature of the state space to be

$$R = \frac{\pi\sqrt{qp}}{24\left(\pi\sqrt{qp}-3\right)^3}\left(\pi^2 qp - 9\pi\sqrt{qp} + 18\right) \tag{5.8}$$

In the large charge limit for which the asymptotic expansion for the macroscopic entropy is valid, the scalar curvature is regular but vanishingly small.

Although typically thermodynamic geometry, particularly Ruppeiner geometry, is based on the macroscopic thermodynamic entropy an implicit assumption of a statistical basis in the framework of a canonical ensemble is involved. Given the mismatch between the duality invariant macroscopic entropy corrected by non holomorphic modifications of the generalized prepotential and the microscopic entropy computed in the dual heterotic picture from degeneracy of appropriate BPS states, it is instructive to explore also the thermodynamic geometry based on the microscopic entropy expression. Although this is obtained from a mixed canonical-microcanonical ensemble in the OSV picture involving a topological string framework, a comparison of the scalar curvatures of the thermodynamic geometry based on the two different entropy expressions may lead to certain physical insight.

The microscopic entropy of the two charged small black hole from the degeneracy of perturbative BPS heterotic string states is given as

$$S(q,p) = 4\pi\sqrt{qp} - \frac{27}{4}\ln(qp) \tag{5.9}$$

The Ruppeiner metric for the equilibrium state space of the system based on this entropy is given by the Hessian matrix of the entropy with respect to the charges to be

$$ds^2 = \left(\frac{\pi}{q}\sqrt{\frac{p}{q}} - \frac{27}{4q^2}\right)dq^2 - \frac{2\pi}{\sqrt{qp}}dqdp + \left(\frac{\pi}{p}\sqrt{\frac{q}{p}} - \frac{27}{4p^2}\right)dp^2 \tag{5.10}$$

The determinant of the metric tensor is $g = -\frac{27}{16q^2p^2}(8\pi\sqrt{qp} - 27)$. Notice that once again the determinant is non zero and negative for the large charge limit being considered here and we obtain a non degenerate Ruppeiner geometry even at extremality. The scalar curvature of the state space in the microscopic picture may now easily computed to be

$$R = \frac{16\pi\sqrt{qp}}{27\left(8\pi\sqrt{qp} - 27\right)^3}\left(32\pi^2 qp - 324\pi\sqrt{qp} + 729\right) \tag{5.11}$$

The scalar curvature is finite and regular in the large charge limit although vanishingly small. This indicates as usual a stable thermodynamic system with a non interacting statistical basis. Although the two scalar curvatures are different in their exact expressions their overall behavior is similar indicating that the microscopic and the macroscopic pictures are in close conformity.

5.3 Thermodynamic geometry and higher α' corrections to small black holes

It is possible to incorporate further higher derivative corrections to the microscopic entropy for the two charged small black holes in Type IIA string theory compactified on $K_3 \times T^2$. It turns out that the first subleading order and the logarithmic corrections are identical to the ones considered before and to the other subleading orders we have the corrected microscopic entropy as a series

$$S_{\text{micro}} = 4\pi\sqrt{qp} - \frac{27}{4}\ln qp + \frac{15}{2}\ln 2 - \frac{675}{32\pi\sqrt{qp}} - \frac{6075}{2048\pi^2 qp} + \cdots \tag{5.12}$$
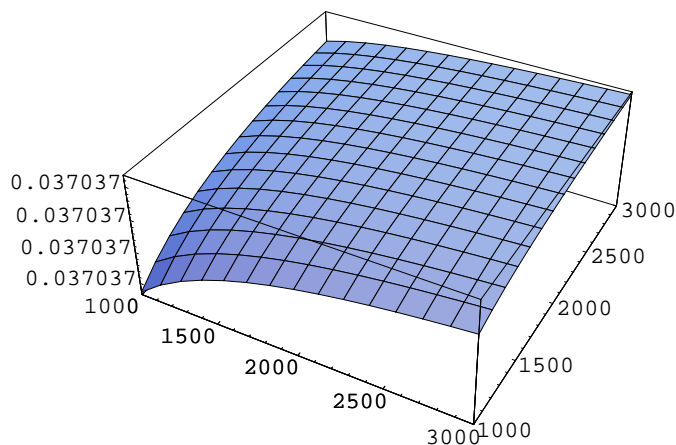
Figure 3: Ruppeiner curvature for the small black hole with $\frac{1}{\sqrt{qp}}$ corrections, as a function of q and p.

It is now possible to study the modification to the thermodynamic geometry arising from the effect of the $\frac{1}{\sqrt{qp}}$ term through the Hessian matrix of the entropy with respect to the charges. This provides the corrected Ruppeiner metric as

$$ds^2 = \left(\frac{\pi}{q}\sqrt{\frac{p}{q}} - \frac{27}{4q^2} + \frac{2025}{128\pi q^2\sqrt{qp}}\right) dq^2 - 2\left(\frac{\pi}{\sqrt{qp}} - \frac{675}{128\pi(qp)^{3/2}}\right) dqdp$$
$$+ \left(\frac{\pi}{p}\sqrt{\frac{q}{p}} - \frac{27}{4p^2} + \frac{2025}{128\pi p^2\sqrt{qp}}\right) dp^2 \tag{5.13}$$

The determinant of the metric tensor is

$$g = -\frac{27}{2048\pi^2 q^3 p^3}(1024\pi^3 qp\sqrt{qp} - 6656\pi^2 qp + 16200\pi\sqrt{qp} - 16875) \tag{5.14}$$

and is non zero in the large charge limit and we have non degenerate thermodynamic geometry at extremality. Notice that the determinant is negative suggesting an indefinite signature for the Riemannian line element in the state space and our earlier arguments continue to remain valid. The scalar curvature of the state space may now be computed from the Ruppeiner metric. The exact expression for the scalar curvature is lengthy, and we present it in the appendix. In figure 3, we present the behavior of the scalar curvature in the large charge limit graphically, as a function of q and p.

It is instructive further to examine whether the other subleading terms significantly modify the thermodynamic geometry of the equilibrium state space. To this end we consider next the effect of the $\frac{1}{qp}$ contributions in eq. (5.12) to the entropy on the state space geometry and the scalar curvature. The Ruppeiner metric, in this case, is given by

$$ds^2 = \left(\frac{\pi}{q}\sqrt{\frac{p}{q}} - \frac{27}{4q^2} + \frac{2025}{128\pi q^2\sqrt{qp}} - \frac{6075}{1024\pi^2 q^3 p}\right) dq^2$$
$$-2\left(\frac{\pi}{\sqrt{qp}} - \frac{675}{128\pi(qp)^{3/2}} - \frac{6075}{2048\pi^2 q^2 p^2}\right) dqdp$$
$$+ \left(\frac{\pi}{p}\sqrt{\frac{q}{p}} - \frac{27}{4p^2} + \frac{2025}{128\pi p^2\sqrt{qp}} + \frac{6075}{1024\pi^2 qp^3}\right) dp^2 \tag{5.15}$$
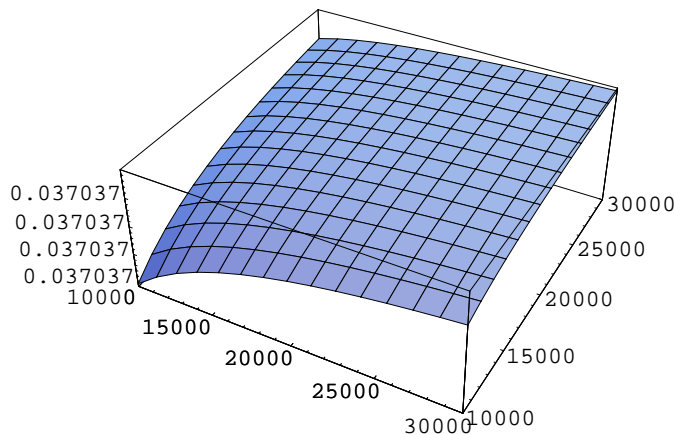
– 19 –

Figure 4: Ruppeiner curvature with $\frac{1}{qp}$ corrections, as a function of q and p.

The determinant of the metric is given as

$$g = -\frac{27}{4194304\pi^4 q^4 p^4}\left(2097152\pi^5 p^2 q^2\sqrt{qp} + 30412800\pi^3 pq\sqrt{qp}\right.$$
$$\left. - 13631488\pi^4 q^2 p^2 - 22118400\pi^2 pq - 24300000\pi\sqrt{qp} - 4100625\right) \quad (5.16)$$

and is nonzero in the large charge limit and hence defines a non degenerate thermodynamic geometry at extremality. It is now possible to compute the thermodynamic scalar curvature and compare with the scalar curvature of the state space due to corrections upto order $\frac{1}{\sqrt{qp}}$. The exact expression is provided in the appendix. In figure 4, we present a graphical analysis of the same. The curvature is regular in the large charge limit but vanishingly small just as for the other examples that we have considered. It is observed that the modification to the scalar curvature due to the inclusion of the $\frac{1}{qp}$ correction to the entropy is marginal.

5.4 String loop corrections and exact entropy expression

Apart from higher derivative terms in the low energy effective supergravity action there are higher derivative contributions to the macroscopic entropy also from purely quantum corrections arising from perturbative string loop corrections and non perturbative contributions. Hence at the R^4 level it is necessary to also consider $\sqrt{\frac{q}{p}}$ contributions to the macroscopic entropy arising from string loop corrections. For the two charged small black hole it is possible to establish the string loop corrected entropy expression from application of the Sen entropy function formalism and scaling arguments. An explicit analysis leads to [27]

$$S_{\mathrm{BH}} = \sqrt{aqp + bq} \quad (5.17)$$

with $q \gg p \gg 1$, where b is a constant depending on the loop corrections and a is an arbitrary constant. Invoking T duality invariance modifies the entropy expression to

$$S_{\mathrm{BH}} = \sqrt{aqp + b\left(q + p\right)} \quad (5.18)$$

A modified microscopic state counting for the two charged small black hole from the degeneracy of BPS states in the dual heterotic string picture leads to the microscopic entropy $S_{\text{micro}} = \sqrt{16qp + \frac{2}{3}q}$ which is consistent with the T duality invariant expression for the macroscopic entropy. A similar expression from microscopic state counting also arises from $D_0 D_4$ black holes in Type IIA string theory compactified on $K_3 \times T^2$. These possess near horizon geometry of $AdS_2 \times S^2 \times CY_3$ where $CY_3 \simeq K_3 \times T^2$. Quite surprisingly Linde and Kallosh [28] has proposed an exact expression for the entropy from quantum information theory as

$$S_{\text{BH}} \sim \sqrt{aqp + b(q + p)} \tag{5.19}$$

which is also consistent with the T duality invariant expression.

It is instructive to study the thermodynamic geometry of the equilibrium state space of the two charged small black holes, implied by the exact entropy expressions. The Ruppeiner metric for the state space from the Hessian matrix of the exact entropy expression is given as;

$$ds^2 = \frac{(ap + b)^2}{4 \left(aqp + b(p + q)\right)^{3/2}} dq^2 + \frac{(aq + b)^2}{4 \left(aqp + b(p + q)\right)^{3/2}} dp^2$$
$$+ \left(\frac{(aq + b)(ap + b)}{2 \left(aqp + b(p + q)\right)^{3/2}} - \frac{a}{\left(aqp + b(p + q)\right)^{1/2}} \right) dq dp \tag{5.20}$$

The determinant of the metric is $g = \frac{ab^2}{4(aqp+b(p+q))^2}$ and is non zero everywhere and hence we have a non degenerate thermodynamic geometry of the equilibrium state space of the two charged extremal small black hole. The scalar curvature of the state space may now be computed from the Ruppeiner metric to be

$$R = -\frac{a^2 qp + abq + abp + b^2}{b^2 \sqrt{aqp + bq + bp}}. \tag{5.21}$$

Observe that the scalar curvature is regular but vanishingly small once again. As usual this indicates a stable thermodynamic system with a non interacting microscopic statistical basis. The scalar curvature of the state space geometry following from the exact entropy expression is negative.

6. Summary and conclusions

In this paper, we have applied, for the first time, the formalism of thermodynamic geometries to degenerate quantum ground states at zero temperatures, exemplified by extremal black holes in Type II string theories. Although alluded to earlier in [10] there was no conclusive computation for equlibrium state space geometry of extreml black holes. Such systems exhibiting macroscopic degeneracies are well known in the physics of condensed matter like spin glasses. As stated in the paper, the non zero entropy of extremal black holes suggests a limiting characterisation of conventional thermodynamics to degenerate quantum ground states at zero temperatures. Our construction is a geometrical realization

of the equilibrium thermodynamic state space of such systems typified by extremal black holes which exhibit macroscopic degeneracy at zero temperatures.

It is well known that black hole solutions in $N \geq 2$ supergravity involves moduli spaces with special Kähler geometry. In particular, they exhibit an attractor phenomena as a consequence of which moduli fields flow under radial evolution to fixed values in terms of the charges at the horizon which is a fixed point of the flow. The entropy is thus a function of the charges only, and independent of the asymptotic values of the moduli ensuring the validity of an underlying microscopic statistical basis in terms of fundamental string states or D-brane systems. The present investigation serves as a prelude to explore the thermodynamics of extremal black holes in Type II string theories, away from the attractor fixed point and a consequent geometrical understanding of the attractor mechanism and the attractor fixed point as possible restrictions in the equilibrium state space extended by the moduli variables.

Quite obviously our construction is a radical departure from the original domain of application of the formalism of thermodynamic geometries to conventional thermodynamic systems at finite non-zero temperatures. Hence certain modifications of the scope of the formalism was naturally expected. In particular the absence of classical fluctuations of a thermal origin at zero temperatures rules out the connection between the probability distribution of classical fluctuations and a positive definite invariant Riemannian form over the equilibrium state space. In fact our construction clearly illustrates that although a non degenerate thermodynamic geometry emerges, the signature of the invariant Riemannian form is indefinite and may also be sensitive to the higher derivative corrections to the macroscopic entropy. This follows clearly from the fact that the Hessian of the ADM mass w.r.t. the scalar moduli is proportional to the moduli space metric at the attractor fixed point with the BPS mass as the proportionality constant. For the extremal black holes at the two derivative level the BPS mass is equal to the macroscopic entropy. Thus inverting this relation, the sign of the Hessian of the entropy w.r.t. the conserved charges clearly depends on the signature of the moduli space metric. Higher derivative corrections however may possibly modify this relation in which the signature of the Riemanian line element in the state space of extremal black holes would be sensitive to the higher derivative contributions to the entropy. Given the connection between the scalar curvature and an interacting statistical basis for non extremal black holes, the same should remain valid also for extremal black holes. The divergences of the scalar curvature in the case of extremal black holes may then possibly describe quantum phase transitions between distinct vacua in the moduli space. The attractor mechanism and in particular the study of Ferrara et al. [10] suggests a role of the moduli space in determining the signature of the invariant interval over the state space and curvature singularities with phase transitions amongst vacua in the moduli space. This is an ongoing issue under our investigation and would be reported in the future.

In the present investigation, we have studied three diverse examples of extremal charged black hole solutions of Type II supergravity which arise as low energy limits of string compactifications on Calabi-Yau manifolds. These are four charged black holes in $D = 4$ and $D = 10$ described by D2-D6-NS5-P brane system, three charged black holes in $D = 5$ and

$D = 10$ described by D1-D5-P system and two charged small black holes in Type II string theories. Quite remarkably, we have found that in spite of conventional thermodynamic notions being invalid at extremality, the equilibrium state space in all these examples admits of a non degenerate Riemannian geometric structure. Although the signature of the invariant interval over the state space is no more positive definite and as argued depends on the moduli space metric and the higher derivative constributions. As discussed earlier, this is not an issue since classical fluctuations are necessarily absent at zero temperatures although quantum fluctuations may be present. We have further computed the scalar curvatures in all these cases and it was found that the scalar curvatures were regular everywhere but vanishingly small in the limit of large charges in which the asymptotic expressions for the entropy were valid. This indicated a non interacting microscopic statistical basis. The absence of divergences in the scalar curvatures implied that the solution were thermodynamically stable and no phase transitions were evident. The modification of the geometry and the scalar curvature due to higher derivative contributions to the macroscopic entropy were also explored. We found that the scalar curvatures were only marginally sensitive to the higher derivative modifications of the entropy. One interesting aspect of our study was that for the case of the two charged small black holes a non degenerate thermodynamic geometry required the inclusion of non holomorphic terms which renders the entropy to be duality invariant. A study of the state space geometry away from the attractor fixed point may possibly clarify this remarkable connection. This is under investigation and we hope to report on this issue in the near future. Furthermore, we also studied the thermodynamic state space based on an exact entropy expression arsing from a quantum information theoretic perspective. The thermodynamic geometry based on this entropy was also found to be non degenerate but now with a positive definite metric. The scalar curvature was regular but small in the limit of large charges and negative. We mention in this connection that we have also investigated two other examples, that of a charged extremal dyonic black hole and the case of a rotating attractor black hole. We found that at the two derivative level the corresponding state space geometries are either degenerate or the Ruppeiner metric is identically zero. For the rotating black hole the extremal limit is also a Davies phase transition point and the zero determinant simply signifies this. Quite obviously from the examples we have studied it seems that higher derivative contributions would possibly also lead to non degenerate thermodynamic geometries in these cases. As we have mentioned our construction is a prelude to studying the state space geometry away from the attractor fixed point where the state space is now extended by the scalar moduli as thermodynamic variables. Such a study is expected to provide an understanding of the attractor mechanism in relation to thermodynamic state space geometries and a geometrical comprehension of the attractor fixed point. We hope to report on this exciting issue in the future.

Acknowledgments

A. Explicit forms for some state space scalar curvatures

In this appendix we provide the explicit forms of the various scalar curvatures for the Ruppeiner geometries of the thermodynamic state spaces of the examples of extremal charged black holes that we have considered, including higher derivative corrections.

For the $D = 10$ black holes described by the $D2 - D6 - NS5 - P$ systems the scalar curvature with higher derivative corrections is

$$
\begin{aligned}
R = \frac{3N_2 N_6 \sqrt{N_p} N_5^{5/2}}{4\pi} & (6C^2 \sqrt{N_p} N_2^3 N_5^6 N_6^3 + 5C N_2^4 N_5^{17/2} N_6^4 \sqrt{N_2 N_5 N_6 N_p} \\
& + 50C N_2 N_5^{5/2} N_6 \sqrt{N_2 N_5 N_6 N_p} + N_2^6 \sqrt{N_p} N_5^{12} N_6^6 + 100C^4 \sqrt{N_p})^{-3} \\
& (17C^2 N_2^{13} N_6^{13} N_5^{53/2} \sqrt{N_2 N_5 N_6 N_p} + 3837C^4 N_2^{10} N_6^{10} N_5^{41/2} \sqrt{N_2 N_5 N_6 N_p} \\
& + 565000C^9 \sqrt{N_p} N_2^3 N_5^6 N_6^3 - 620000C^{10} N_2 N_5^{5/2} N_6 \sqrt{N_2 N_5 N_6 N_p} \\
& - 114C^3 \sqrt{N_p} N_2^{12} N_5^{24} N_6^{12} + 26C \sqrt{N_p} N_2^{15} N_5^{30} N_6^{15} \\
& + 47472C^6 N_2^7 N_5^{29/2} N_6^7 \sqrt{N_2 N_5 N_6 N_p} - 800000C^{11} \sqrt{N_p} \\
& + 604500C^8 N_2^4 N_5^{17/2} N_6^4 \sqrt{N_2 N_5 N_6 N_p} + 2N_2^{16} N_5^{65/2} N_6^{16} \sqrt{N_2 N_5 N_6 N_p} \\
& + 20280C \sqrt{N_p} N_2^9 N_5^{18} N_6^9 + 178980C^7 \sqrt{N_p} N_2^6 N_5^{12} N_6^6)
\end{aligned} \tag{A.1}
$$

where the constant C has been defined in the main text.

For the $D = 10$ extremal black holes described by D1-D5-P brane system we have the scalar curvature of the state space corrected by higher derivative R^4 contributions arising from the string three loop scattering amplitude is given as

$$
\begin{aligned}
R = \frac{3}{4} \sqrt{N_1 N_5 N_p} \sqrt{N_1 N_5} \Big(& \frac{9N_1^{10} N_5^{10} b \sqrt{N_1 N_5} - 256 N_1^7 N_5^7 b^3 \sqrt{N_1 N_5}}{\pi N_p (6N_1 N_5 \sqrt{N_1 N_5} b^2 - 20 b^3 + N_1^4 N_5^4 \sqrt{N_1 N_5})^3} \\
& - \frac{-1920 N_1 N_5 b^7 \sqrt{N_1 N_5} + 1700 N_1^4 N_5^4 b^5 \sqrt{N_1 N_5} + N_1^{12} N_5^{12}}{\pi N_p (6N_1 N_5 \sqrt{N_1 N_5} b^2 - 20 b^3 + N_1^4 N_5^4 \sqrt{N_1 N_5})^3} \\
& + \frac{6520 N_1^3 N_5^3 b^6 - 1600 b^8 - 48 N_1^6 N_5^6 b^4 + 8 N_1^9 N_5^9 b^2}{\pi N_p (6N_1 N_5 \sqrt{N_1 N_5} b^2 - 20 b^3 + N_1^4 N_5^4 \sqrt{N_1 N_5})^3} \Big)
\end{aligned} \tag{A.2}
$$

Next, we provide the Ruppeiner scalar curvature of the state space geometry based on the microscopic entropy expression for two charged small black holes in Type II string theory compactified on $K_3 \times T^2$ corrected by $\frac{1}{\sqrt{qp}}$ contributions:

$$
\begin{aligned}
R = \frac{1}{8192 n^2 w^2 \sqrt{qp} \pi} \big[& (-265420800\pi^3 qp\sqrt{qp} - 29360128\pi^5 q^2 p^2 \sqrt{qp} \\
& + 5948640000\pi \sqrt{qp} - 7688671875 + 2097152 q^3 p^3 \pi^6 + 160350208 q^2 p^2 \pi^4 \\
& - 1149033600 qp\pi^2)(1024\pi^3 qp\sqrt{qp} - 6656 qp\pi^2 + 16200\pi\sqrt{qp} - 16875)^{-3} \big]
\end{aligned}
$$

Finally, the Ruppeiner curvature for the two charged small black hole, with the $\frac{1}{qp}$ corrections is

$$R = \frac{65536\pi^3 qp\sqrt{qp}}{27}(2097152\pi^5 p^2 q^2\sqrt{qp} + 30412800\pi^3 pq\sqrt{qp}$$
$$-13631488\pi^4 q^2 p^2 - 22118400\pi^2 pq - 24300000\pi\sqrt{qp} - 4100625)^{-3}$$
$$(22823730196875000\pi\sqrt{qp} + 465892154818560000\pi^5 p^2 q^2\sqrt{qp}$$
$$-201080268288000000\pi^3 pq\sqrt{qp} - 192782236070707200\pi^7 q^3 p^3\sqrt{qp}$$
$$+3783403212890625 - 95786360635392000\pi^9 q^4 p^4\sqrt{qp}$$
$$-2744381022928896\pi^{11} q^5 p^5\sqrt{qp} - 177903493447680000\pi^6 q^3 p^3$$
$$+22282702648508416\pi^{10} q^5 p^5 + 218241931463884800\pi^8 q^4 p^4$$
$$+140737488355328\pi^{12} q^6 p^6 - 9486221850000000\pi^2 pq$$
$$-42628654080000000\pi^4 q^2 p^2) \tag{A.3}$$

References

[1] P.K. Townsend, *Black holes*, `gr-qc/9707012`.

[2] R.M. Wald, *Black hole entropy is the Noether charge*, *Phys. Rev.* **D 48** (1993) 3427 [`gr-qc/9307038`];
V. Iyer and R.M. Wald, *Some properties of Noether charge and a proposal for dynamical black hole entropy*, *Phys. Rev.* **D 50** (1994) 846 [`gr-qc/9403028`];
R.M. Wald, *Gravitation, thermodynamics and quantum theory*, *Class. Quant. Grav.* **16** (1999) A177 [`gr-qc/9901033`].

[3] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Examples of stationary BPS solutions in $N = 2$ supergravity theories with R^2-interactions*, *Fortschr. Phys.* **49** (2001) 557 [`hep-th/0012232`];
T. Mohaupt, *Black hole entropy, special geometry and strings*, *Fortschr. Phys.* **49** (2001) 3 [`hep-th/0007195`];
B. de Wit, *Introduction to black hole entropy and supersymmetry*, `hep-th/0503211`;
B. Pioline, *Lectures on on black holes, topological strings and quantum attractors*, *Class. and Quant. Grav.* **23** (2006) S981 [`hep-th/0607227`];
A. Dabholkar, F. Denef, G.W. Moore and B. Pioline, *Precision counting of small black holes*, *JHEP* **10** (2005) 096 [`hep-th/0507014`];
A. Sen, *Black Hole Entropy Function, Attractors and Precision Counting of Microstates*, `arXiv:0708.1270`;
T. Mohaupt, *Strings, higher curvature corrections and black holes*, `hep-th/0512048`.

[4] L. Tisza, *Generalized thermodynamics*, Pub. MIT Press, Cambridge, MA (1966);
H.B. Callen, *Thermodynamics and an introduction to thermostatistics*, Pub. Wiley, New York (1985).

[5] F. Weinhold, *Metric geometry of equilibrium thermodynamics*, *J.Chem. Phys.* **63** (1975) 2479;
Metric geometry of equilibrium thermodynamics. II. Scaling, homogeneity, and generalized Gibbs-Duhem relations, *J. Chem. Phys.* **63** (1975) 2484.

[6] G. Ruppeiner, *Riemannian geometry in thermodynamic fluctuation theory*, *Rev. Mod. Phys.* **67** (1995) 605 [*Erratum ibid.* **68** (1996) 313];
M. Santoro and A.S. Benight, *On the geometrical thermodynamics of chemical reactions*, `math-ph/0507026`.

[7] K. Huang, *Statistical mechanics*, Publishers John Wiley (1963);
L.D. Landau and E.M. Lifshitz, *Statistical physics*, Publishers Pergamon (1969).

[8] G. Ruppeiner, *Thermodynamics: a Riemannian geometric model*, *Phys. Rev.* **A 20** (1979) 1608; *Thermodynamic Critical Fluctuation Theory?*, *Phys. Rev. Lett.* **50** (1983) 287; *New thermodynamic fluctuation theory using path integrals*, *Phys. Rev.* **A 27** (1983) 1116; *Thermodynamic curvature of the multicomponent ideal gas*, *Phys. Rev.* **A 41** (1990) 2200.

[9] G. Ruppeiner, *Stability and fluctuations in black hole thermodynamics*, *Phys. Rev.* **D 75** (2007) 024037.

[10] S. Ferrara, G.W. Gibbons and R. Kalloh, *Black holes and critical points in moduli space*, *Nucl. Phys.* **B 500** (1997) 75 [hep-th/9702103].

[11] J.-Y. Shen, R.-G. Cai, B. Wang and R.-K. Su, *Thermodynamic geometry and critical behavior of black holes*, *Int. J. Mod. Phys.* **A 22** (2007) 11 [gr-qc/0512035].

[12] J.E. Aman, I. Bengtsson and N. Pidokrajt, *Flat information geometries in black hole thermodynamics*, *Gen. Rel. Grav.* **38** (2006) 1305 [gr-qc/0601119];
J.E. Aman and N. Pidokrajt, *Geometry of higher-dimensional black hole thermodynamics*, *Phys. Rev.* **D 73** (2006) 024017 [hep-th/0510139];
J. Aman, I. Bengtsson and N. Pidokrajt, *Geometry of black hole thermodynamics*, *Gen. Rel. Grav.* **35** (2003) 1733 [gr-qc/0304015];
G. Arcioni and E. Lozano-Tellechea, *Stability and critical phenomena of black holes and black rings*, *Phys. Rev.* **D 72** (2005) 104021 [hep-th/0412118].

[13] T. Sarkar, G. Sengupta and B. Nath Tiwari, *On the thermodynamic geometry of BTZ black holes*, *JHEP* **11** (2006) 015 [hep-th/0606084].

[14] A. Sen, *Black holes, elementary strings and holomorphic anomaly*, *JHEP* **07** (2005) 063 [hep-th/0502126].

[15] A. Sen, *Strong - weak coupling duality in four-dimensional string theory*, *Int. J. Mod. Phys.* **A 9** (1994) 3707 [hep-th/9402002]; *Black hole entropy function and the attractor mechanism in higher derivative gravity*, *JHEP* **09** (2005) 038 [hep-th/0506177];
B. Sahoo and A. Sen, *BTZ black hole with Chern-Simons and higher derivative terms*, *JHEP* **07** (2006) 008 [hep-th/0601228]; *Higher derivative corrections to non-supersymmetric extremal black holes in N = 2 supergravity*, *JHEP* **09** (2006) 029 [hep-th/0603149];
A. Sen, *Entropy function for heterotic black holes*, *JHEP* **03** (2006) 008 [hep-th/0508042];
D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [hep-th/0510147];
J.R. David and A. Sen, *CHL dyons and statistical entropy function from D1 − D5 system*, *JHEP* **11** (2006) 072 [hep-th/0605210].

[16] A. Dabholkar, N. Iizuka, A. Iqubal, A. Sen and M. Shigemori, *Spinning strings as small black rings*, *JHEP* **04** (2007) 017 [hep-th/0611166];
A. Dabholkar, A. Sen and S.P. Trivedi, *Black hole microstates and attractor without supersymmetry*, *JHEP* **01** (2007) 096 [hep-th/0611143];
D. Astefanesei, K. Goldstein, R.P. Jena, A. Sen and S.P. Trivedi, *Rotating attractors*, *JHEP* **10** (2006) 058 [hep-th/0606244];
A. Sen, *Stretching the horizon of a higher dimensional small black hole*, *JHEP* **07** (2005) 073 [hep-th/0505122];
D. Astefanesei, K. Goldstein and S. Mahapatra, *Moduli and (un)attractor black hole thermodynamics*, *Gen. Rel. Grav.* **40** (2008) 2069 [hep-th/0611140].

[17] S. Minwalla, *Black holes in large-N gauge theories*, *Class. and Quant. Grav.* **23** (2006) S927.

[18] S. Minwalla, private communication.

[19] V. Balasubramanian, J. de Boer, V. Jejjala and J. Simon, *The library of Babel: on the origin of gravitational thermodynamics*, *JHEP* **12** (2005) 006 [hep-th/0508023].

[20] P.J. Silva, *Thermodynamics at the BPS bound for black holes in AdS*, *JHEP* **10** (2006) 022 [hep-th/0607056]; *Phase transitions and statistical mechanics for BPS black holes in AdS/CFT*, *JHEP* **03** (2007) 015 [hep-th/0610163];
O.J.C. Dias and P.J. Silva, *Euclidean analysis of the entropy functional formalism*, *Phys. Rev.* **D 77** (2008) 084011 [arXiv:0704.1405].

[21] R. Balbinot, A. Fabbri, S. Farese and R. Parentani, *Hawking radiation from extremal and non-extremal black holes*, *Phys. Rev.* **D 76** (2007) 124010 [arXiv:0710.0388].

[22] G.W. Gibbons, R. Kallosh and B. Kol, *Moduli, scalar charges and the first law of black hole thermodynamics*, *Phys. Rev. Lett.* **77** (1996) 4992 [hep-th/9607108].

[23] J.M. Maldacena, A. Strominger and E. Witten, *Black hole entropy in M-theory*, *JHEP* **12** (1997) 002 [hep-th/9711053].

[24] R.-G. Cai and D.-W. Pang, *Entropy function for 4-charge extremal black holes in type IIA superstring theory*, *Phys. Rev.* **D 74** (2006) 064031 [hep-th/0606098].

[25] A. Ghodsi, R^4 *corrections to D1D5p black hole entropy from entropy function formalism*, *Phys. Rev.* **D 74** (2006) 124026 [hep-th/0604106].

[26] M.J. Duff, *Electric/magnetic duality and its stringy origins*, *Int. J. Mod. Phys.* **A 11** (1996) 4031 [hep-th/9509106].

[27] A. Sinha and N.V. Suryanarayana, *Two-charge small black hole entropy: string-loops and multi-strings*, *JHEP* **10** (2006) 034 [hep-th/0606218]; *Extremal single-charge small black holes: entropy function analysis*, *Class. and Quant. Grav.* **23** (2006) 3305 [hep-th/0601183];
A. Sinha, *Small black holes in string theory: some puzzles*, Caltech Lecture (2006).

[28] R. Kallosh and A. Linde, *Strings, black holes and quantum information*, *Phys. Rev.* **D 73** (2006) 104033 [hep-th/0602061].